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On exponential Yang–Mills connections

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Abstract

In this paper, we study a new functional, i.e., the exponential Yang–Mills functional $\mathscr{Y}\mathscr{M}_e$ on the space of all smooth connections ∇ of a vector bundle E over a compact Riemannian manifold (M, g) which is defined by

$$\mathscr{Y}\mathscr{M}_e(\nabla) = \int_M \exp(\tfrac{1}{2}\|R^\nabla\|^2) v_g ,$$

where $\|R^\nabla\|$ is the curvature tensor of a connection ∇. A critical point of $\mathscr{Y}\mathscr{M}_e$ is called an exponential Yang–Mills connection. If $\|R^\nabla\|$ is constant, a smooth connection ∇ is an exponential Yang–Mills connection if it is a Yang–Mills one. We show for any vector bundle E, that the functional $\mathscr{Y}\mathscr{M}_e$ admits a minimising connection ∇ which is C^α-Hölder continuous for all $0<\alpha<1$. We show the existence theorem of a smooth exponential Yang–Mills connection and study its properties and the second variation formula.

Keywords: Exponential Yang–Mills connection; Exponential Yang–Mills functional; Conformal
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1. Introduction and statement of results

The purpose of this paper is to set up a frame work on a new functional, i.e., the exponential Yang–Mills functional, in the calculus of variation on the space of connections of a vector bundle over a compact Riemannian manifold.

Before beginning to state our results, let us recall recent results on harmonic maps and exponentially harmonic maps due to Eells and Lemaire [EL], Eells and Ferreira [EF] and Hong [H]. Eells and Lemaire [EL] considered exponentially harmonic maps instead of harmonic maps. Let (M, g) and (N, h) be two compact

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Riemannian manifolds and $\varphi : M \to N$ a smooth map. Harmonic maps are extremals of the energy functional

$$E(\varphi) = \int_M e(\varphi) v_g ,$$

where $e(\varphi) = \frac{1}{2} |\mathrm{d}\varphi|^2$ is the energy density and v_g is the canonical volume element. The map φ is harmonic if and only if it satisfies the Euler–Lagrange equation

$$\tau(\varphi) = \mathrm{div}(\mathrm{d}\varphi) = 0 .$$

The existence problem for harmonic maps is the following: given two Riemannian manifolds (M, g), (N, h) and a homotopy class $\mathcal{H}$ of smooth maps from M to N, when is there a harmonic map in $\mathcal{H}$?

This problem has been studied extensively, and the answer depends on the manifolds and the homotopy class. To obtain existence of solutions in all dimensions, without conditions on the manifolds, Eells and Lemaire [EL] considered another problem of the calculus of variations. Namely, they defined the *exponential energy* of φ as

$$E_{\mathrm{e}}(\varphi) = \int_M \exp(\tfrac{1}{2} |\mathrm{d}\varphi|^2) v_g$$

and called a smooth extremal of E_{e} an *exponentially harmonic map*. They showed the following theorem and interesting properties of exponentially harmonic maps:

Theorem 1.1. (Eells and Lemaire [EL]). *Let (M, g) and (N, h) be two compact Riemannian manifolds, $\mathcal{H}$ a homotopy class of smooth maps from M to N. Then $\mathcal{H}$ contains an E_{e}-minimising map, which is α-Hölder continuous for all $0 < \alpha < 1$.*

Note that, if $e(\varphi)$ is constant, a smooth map φ is harmonic iff it is exponentially harmonic. For the existence of harmonic maps and exponentially harmonic maps, it is known that:

Theorem 1.2. *If $\dim M \geq 3$, then for any homotopy class $\mathcal{H}$,*

(1) (Eells and Ferreira [EF]) there exist a C^∞ Riemannian metric $\tilde{g}$ on M conformal to g and a C^∞ map φ in $\mathcal{H}$ such that $\varphi : (M, \tilde{g}) \to (N, h)$ is harmonic;

(2) (Hong [H]) there exist a C^∞ Riemannian metric g' on M conformal to g and a C^∞ map ψ in $\mathcal{H}$ such that $\psi: (M, g') \to (N, h)$ is exponentially harmonic.

Now we start the set up of our exponential Yang–Mills connections. It is well known that both theories of Yang–Mills connections and harmonic maps have certain strong similarities. The existence problem for Yang–Mills connections is the following: given a compact Riemannian manifold (M, g) and a G-vector bundle E to a compact Lie group G over M, when is there a Yang–Mills connection on E?

This problem has also been studied and it turns out that the dimension of M plays an essential role. For harmonic maps of (M, g) into (N, h), $\dim M = 2$ is critical. But for Yang–Mills connections, the critical dimension of M is four. In the case $\dim M = 4$, special solutions are well known. Namely, a connection ∇ of E is *self-dual* (resp. *anti-self-dual*) if its curvature tensor R^∇ satisfies $*R^\nabla = R^\nabla$ (resp. $*R^\nabla = -R^\nabla$) where $*$ is the Hodge star operator on exterior 2-forms on M. (Anti-)self-dual connections are known to exist and the moduli space of such connections on a 4-manifold M influences the topology of M (cf. Donaldson [D1]). In the case $\dim M = 2$ or 3, the moduli space of Yang–Mills connections also plays important rolls (cf. Atiyah and Bott [AB], Floer [F]). For the higher dimensional case, if (M, g) is a compact Kähler manifold, a stable vector bundle over M admits a unique special Yang–Mills connection, called the Einstein–Hermitian connection, which is a natural extension of (anti-)self-dual connection (cf. Kobayashi [K], Donaldson [D2], Uhlenbeck and Yau [UY]). If (M, g) is a strongly pseudo-convex CR manifold, a special Yang–Mills connection is also known (cf. Urakawa [Ur1, Ur2]). All these theories depend on the base manifold (M, g) and a special property of the G-vector bundle E itself, and the existence problem is still unsettled in general.

So in this paper, we introduce and study another problem of the calculus of variations in an analogous way as exponentially harmonic maps in the above [EL]. Namely, we define the *exponential Yang–Mills functional* $\mathscr{Y}\mathscr{M}_{\mathrm{e}}$ in Section 2 as follows. Let (M, g) be a compact Riemannian manifold, E a G-vector bundle over M. Let $\mathscr{C}(E)$ be the space of all C^∞ G-connections of E. For $\nabla \in \mathscr{C}(E)$, let R^∇ be its curvature tensor and define the *exponential Yang–Mills functional* by

$$\mathscr{Y}\mathscr{M}_{\mathrm{e}}(\nabla) = \int_M \exp(\tfrac{1}{2}\|R^\nabla\|^2) v_g .$$

And a smooth extremal of $\mathscr{Y}\mathscr{M}_{\mathrm{e}}$ is called an *exponential Yang–Mills connection*. Note that, if $\|R^\nabla\|$ is constant, then a smooth G-connection ∇ is an exponential Yang–Mills connection if and only if it is a Yang–Mills one. One of our main results is as follows:

Theorem 4.3. *Let (M, g) be a compact Riemannian manifold and E a G-vector bundle over M to a compact Lie group G. Then $\mathscr{Y}\mathscr{M}_{\mathrm{e}}$ admits a minimising connection ∇, which is α-Hölder continuous for all* $0 < \alpha < 1$, *and the norm of which* $\|R^\nabla\|$ *is almost constant.*

For the existence of Yang–Mills connections and exponential Yang–Mills connections, we obtain:

Theorem 1.3. *If* $\dim M \geq 5$, *then for any G-vector bundle E to a compact Lie group G over a compact Riemannian manifold (M, g),* *(1) (cf. Katagiri [KA]) there exist a C^∞ Riemannian metric $\tilde{g}$ on the base manifold M conformal to g and a C^∞ G-connection ∇ on E such that ∇ is a Yang–Mills connection with respect to $\tilde{g}$ (cf. Thm. 5.1);*

(2) there exist a C^∞ Riemannian metric g' on M conformal to g and a C^∞ G-connection ∇' on E such that ∇' is an exponential Yang–Mills connection with respect to g' (cf. Thm. 5.3).

In the case $\dim M = 4$, we obtain:

Theorem 5.6. *If $\dim M = 4$, there exist a C^∞ G-connection ∇ and a C^0 Riemannian metric g' on M conformal to g such that ∇ is an exponential Yang–Mills connection in the weak sense (Section 5) with respect to g'.*

We also calculate the second variational formula of $\mathcal{YM}_e$ and study the precise properties of exponential Yang–Mills connections.

2. The first variation formula

In this section, we prepare several notations and derive the first variation formula of exponential Yang–Mills connections. In this paper, we fix a compact Lie group G, a principal G-bundle P over a compact Riemannian manifold (M, g), and a G-vector bundle over M, $E = P \otimes_\rho \mathbb{R}^r$, associated to P by a faithful representation $\rho: G \to O(r)$. That is, G acts on $P \times \mathbb{R}^r$ by $P \times \mathbb{R}^r \ni (u, y) \mapsto (ut, \rho(t)^{-1} y) \in P \times \mathbb{R}^r$, $t \in G$, whose quotient is denoted by $u \cdot y = [(u, y)] \in P \times_\rho \mathbb{R}^r$. Each element u of P over $x \in M$ defines a linear isomorphism of $\mathbb{R}^r$ onto the fiber E_x, $u: \mathbb{R}^r \to E_x$ by $y \mapsto u \cdot y$. Take a basis $\{e_i\}_{i=1}^r$ of $\mathbb{R}^r$. Any local C^∞ section s of P over an open set U of M induces a local frame $\{s_i\}_{i=1}^r$ of E over U, called a *G-frame*, by $s_i := s \cdot e_i$.

Given a vector bundle F over M, let $\Omega^p(F) = \Gamma(\wedge^p T^*M \otimes F)$ denote the space of all smooth p-forms on M with values in F, $p \geq 0$.

A smooth connection on E is a linear differential operator $\nabla: \Omega^0(E) \to \Omega^1(E)$ such that

$$\nabla(f\sigma) = \mathrm{d}f \otimes \sigma + f \nabla \sigma,$$

for all $f \in C^\infty(M)$, $\sigma \in \Omega^0(E)$. For a G-frame $\{s_i\}_{i=1}^r$ of E on U, by means of

$$\nabla s_i = \sum_{j=1}^r \omega_{ij} s_j,$$

a $r \times r$-matrix valued 1-formed $\omega_U = (\omega_{ij})$, called the connection form of ∇ with respect to $\{s_i\}_{i=1}^r$, is defined on U. Then there exists a unique $r \times r$-matrix valued 1-form ω on P, called the *connection form* of ∇, such that $s^* \omega = \omega_U$. A connection ∇ on E is called a *G-connection* if its connection form ω takes its values in the Lie algebra $\mathfrak{g}$ of G which is identified with a subalgebra of $\mathfrak{gl}(r, \mathbb{R})$ via ρ. This means also that ω_U takes its values in $\mathfrak{g}$ for any G-frame. Let $\mathscr{C}(E)$ be the set of all smooth G-connections ∇ on E.

The group of all automorphisms of E inducing the identity map of M is called the *gauge group*, denoted by $\mathscr{G}(E)$. The gauge group $\mathscr{G}(E)$ is identified with the space of smooth sections of the fiber bundle $P \times_{\mathrm{Ad}} G$, which is the group of all automorphisms φ of P satisfying $\varphi(ua) = \varphi(u)a$, $u \in P$, $a \in G$. The identification is $\mathscr{G}(E) \in \varphi \mapsto \tilde{\varphi}$ where $\tilde{\varphi}(u) := \varphi \circ u$, $u \in P$ considered as a linear isomorphism of $\mathbb{R}^r$ onto E_x. $\mathscr{G}(E)$ acts on $\mathscr{C}(E)$ by

$$\nabla^{\varphi} := \varphi^{-1} \circ \nabla \circ \varphi, \quad \nabla^{\varphi}\sigma := \varphi^{-1}(\nabla(\varphi\sigma)),$$

for $\sigma \in \Omega^0(E)$, $\varphi \in (E)$ and $\nabla \in \mathscr{C}(\mathrm{E})$. ∇^{φ} corresponds to the connection form $\tilde{\varphi}^*\omega$ if ω is the one of ∇. The Lie algebra of $\mathscr{G}(E)$ can be regarded as the space $\Omega^0(P \times_{\mathrm{Ad}} \mathfrak{g})$ of smooth sections of the vector bundle $P \times_{\mathrm{Ad}} \mathfrak{g}$, which is identified with a subbundle of the bundle $\mathrm{End}(E)$ via ρ, denoted by $\mathfrak{g}_E$. The identification is

$$P \times_{\mathrm{Ad}} \mathfrak{g} \ni [(u, A)] \mapsto u \circ \rho(A) \circ u^{-1} \in \mathrm{End}(E).$$

Note that $\mathscr{C}(E)$ admits an affine structure, i.e., the difference of two connections $A = \nabla - \nabla'$ is in $\Omega^1(\mathfrak{g}_E)$ and $\mathscr{C}(E) = \{\nabla + A;\, A \in \Omega^1(\mathfrak{g}_E)\}$ for any fixed element $\nabla \in \mathscr{C}(E)$. Equivalently, the difference of two connection forms $\alpha = \omega - \omega'$ is in $\Omega^1(P \times_{\mathrm{Ad}} \mathfrak{g})$ and all connection forms are obtained by $\omega + \alpha$ with $\alpha \in \Omega^1(P \times_{\mathrm{Ad}} \mathfrak{g})$ for a fixed connection form ω.

To each G-connection ∇ of E, the *curvature tensor* R^{∇} in $\Omega^2(\mathfrak{g}_E)$ is defined by

$$R^{\nabla}(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]},$$

for vector fields X and Y on M. It corresponds to a $\mathfrak{g}$-valued 2-form Ω on P, called the *curvature form* defined by $\Omega := \mathrm{d}\omega + \omega \wedge \omega$, ω being the connection form of ∇. It holds that

$$R^{\nabla^{\varphi}} = \varphi^{-1} \circ R^{\nabla} \circ \varphi,$$

equivalently, $\tilde{\Omega} = \tilde{\varphi}^*\Omega$, $\tilde{\Omega}$ being the curvature form of the connection form $\tilde{\varphi}^*\omega$.

Let $\langle\ ,\ \rangle$ be an inner product on $\mathfrak{g}$ defined by

$$\langle A, B\rangle = -\tfrac{1}{2}\mathrm{tr}(\rho(A)\rho(B)) = \tfrac{1}{2}\mathrm{tr}({}^t\rho(A) \circ \rho(B)), \quad A, B \in \mathfrak{g},$$

which induces a fibre metric metric on $P \times_{\mathrm{Ad}} \mathfrak{g}$. Equivalently it induces a fibre metric on $\mathrm{End}(E)$ by

$$\langle C, D\rangle = \tfrac{1}{2}\mathrm{tr}({}^tC \circ D), \quad C, D \in \mathrm{End}(E_x), x \in M.$$

For a connection ∇ on a vector bundle F over M, let $\mathrm{d}^{\nabla}: \Omega^p(F) \to \Omega^{p+1}(F)$, $p \geq 0$, denote the usual exterior differential operator defined by

$$\mathrm{d}^{\nabla}\omega(X_1 \ldots, X_{p+1}) := \sum_{i=1}^{p+1} (-1)^{i+1} \nabla_{X_i}(\omega(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1})) + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{p+1}),$$

for $\omega \in \Omega^p(F)$ and vector fields $X_1, \ldots, X_{p+1}$ on M. If F admits a fiber metric $\langle\ ,\ \rangle$, define an inner product on $\wedge^p T_x^* M \otimes F_x$ by

$$\langle \psi, \varphi \rangle = \sum_{i_1 < \cdots < i_p} \langle \psi(e_{i_1}, \ldots, e_{i_p}), \varphi(e_{i_1}, \ldots, e_{i_p}) \rangle ,$$

where $\{e_i\}_{i=1}^n$ is an orthonormal basis of T_xM with respect to g. We denote its norm by $\|\ \|$. The global inner product $(\ ,\ )$ on $\Omega^p(F)$ is defined by

$$(\psi, \varphi) = \int_M \langle \psi, \varphi \rangle v_g$$

for $\psi, \varphi \in \Omega^p(F)$. Then we define the operator $\delta^\nabla : \Omega^{p+1}(F) \to \Omega^p(F), p \geq 0$, to be the formal adjoint of the operator d^∇.

It holds by the above definitions that

$$\|R^{\nabla^\varphi}\| = \|R^\nabla\|, \quad \text{or equivalently} \quad \|\tilde{\varphi}^* \Omega\| = \|\Omega\|$$

for $\varphi \in \mathscr{G}(E)$.

Now let us recall definition of the Yang–Mills functional.

Definition 2.1. The *Yang–Mills functional* $\mathscr{YM} : \mathscr{C}(E) \to \mathbb{R}$ is defined by

$$\mathscr{YM}(\nabla) = \frac{1}{2} \int_M \|R^\nabla\|^2 v_g , \tag{2.1}$$

or equivalently, $\mathscr{YM} : \{\text{connection forms } \omega\} \to \mathbb{R}$ by

$$\mathscr{YM}(\omega) = \frac{1}{2} \int_M \|\Omega^\omega\|^2 v_g , \tag{2.1'}$$

Ω^ω being the curvature form of ω.

Now we define the exponential Yang–Mills functional. In the following, we consider only connections on the vector bundle E for simplicity.

Definition 2.2. The *exponential Yang–Mills functional* $\mathscr{YM}_e : \mathscr{C}(E) \to \mathbb{R}$ is defined by

$$\mathscr{YM}_e(\nabla) = \int_M \exp(\tfrac{1}{2}\|R^\nabla\|^2) v_g . \tag{2.2}$$

A critical point $\nabla \in \mathscr{C}(E)$ of the Yang–Mills functional $\mathscr{YM}$ is called a *Yang–Mills connection* and a critical point of the exponential Yang–Mills functional $\mathscr{YM}_e$ is called an *exponential Yang–Mills connection*.

Note that by the above, both $\mathcal{YM}$ and $\mathcal{YM}_{\mathrm{e}}$ are invariant under the action of the gauge group $\mathcal{G}(E)$ on $\mathcal{C}(E)$.

In the following, we calculate the first variation of the functional $\mathcal{YM}_{\mathrm{e}}$. For this, we fix $\nabla \in \mathcal{C}(E)$ and consider a smooth family of G-connections ∇^t, $-\epsilon < t < \epsilon$, such that $\nabla^0 = \nabla$. Write

$$\nabla^t = \nabla + A^t ,$$

where $A^t \in \Omega^1(\mathfrak{g}_E)$ for $|t| < \epsilon$ and $A^0 = 0$. It holds that

$$R^{\nabla^t} = R^{\nabla} + \mathrm{d}^{\nabla} A^t + \tfrac{1}{2}[A^t \wedge A^t] ,$$

where for two φ, $\psi \in \Omega^1(\mathfrak{g}_E)$,

$$[\varphi \wedge \psi](X, Y) = [\varphi(X), \psi(Y)] - [\varphi(Y), \psi(X)], \quad X, Y \in T_x M .$$

Since

$$\begin{aligned} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(\tfrac{1}{2}\|R^{\nabla^t}\|^2) &= \exp(\tfrac{1}{2}\|R^{\nabla}\|^2) \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \tfrac{1}{2}\|R^{\nabla^t}\|^2 \\ &= \exp(\tfrac{1}{2}\|R^{\nabla}\|^2)\langle \mathrm{d}^{\nabla}B, R^{\nabla}\rangle , \\ \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathcal{YM}_{\mathrm{e}}(\nabla^t) &= \int_M \exp(\tfrac{1}{2}\|R^{\nabla}\|^2)\langle \mathrm{d}^{\nabla}B, R^{\nabla}\rangle v_g \\ &= \int_M \langle B, \delta^{\nabla}(\exp(\tfrac{1}{2}\|R^{\nabla}\|^2)R^{\nabla})\rangle v_g , \end{aligned}$$

where

$$B := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \nabla^t \in \Omega^1(\mathfrak{g}_E) .$$

Thus we obtain:

Theorem 2.3. *The first variation of the exponential Yang–Mills functional is given by the formula*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathcal{YM}_{\mathrm{e}} = \int_M \langle B, \delta^{\nabla}(\exp(\tfrac{1}{2}\|R^{\nabla}\|^2)R^{\nabla})\rangle v_g ,$$

where

$$B = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \nabla^t .$$

Consequently, ∇ is an exponential Yang–Mills connection if and only if

$$\delta^{\nabla}(\exp(\tfrac{1}{2}\|R^{\nabla}\|^2)R^{\nabla}) = 0 . \tag{2.3}$$

In particular, if $\|R^{\nabla}\|$ *is constant and* ∇ *is a smooth connection, then* ∇ *is an exponential Yang–Mills connection if and only if it is a Yang–Mills one.*

3. Jensen's inequality and its applications

A function $f: \mathbb{R}^p \to \mathbb{R}$ is called *convex* if for all $x, y \in \mathbb{R}^p$, and $0 \le \lambda \le 1$,

$$f((1-\lambda)x + \lambda y) \le (1-\lambda)f(x) + \lambda f(y) .$$

Then Jensen's inequality is:

Proposition 3.1. *Let* φ *be a convex function on* $\mathbb{R}^p$, *S a set with* $\mu(S) < \infty$, μ *a non-negative bounded measure on S and* $\mathscr{L}^1(S, \mu)$ *the space of all integral measurable functions on S with respect to* μ. *Then for any* $\xi_i \in \mathscr{L}^1(S, \mu)$, $1 \le i \le p$,

$$\varphi(\zeta^1, \ldots, \zeta^p) \le \frac{1}{\mu(S)} \int_S \varphi(\xi_1(x), \ldots, \xi_p(x)) \, \mathrm{d}\mu ,$$

where

$$\zeta^i := \frac{1}{\mu(S)} \int_S \xi_i(x) \, \mathrm{d}\mu .$$

The equality holds if and only if ξ_i *is constant almost everywhere.*

Proof. See [Mo].

To apply this proposition, we prepare the Sobolev space $\mathscr{L}^p_1(E)$ of $\mathscr{L}^p_1$ G-connections, where $\mathscr{L}^p_1$ means the Sobolev space of functions with first derivatives which are p-integrable: We fix ∇^0 as a C^∞ G-connection of E, i.e., $\nabla^0 \in \mathscr{C}(E)$. Let $1 < p < \infty$ and define the Sobolev space by

$$\mathscr{L}^p_1(E) := \{ \nabla = \nabla^0 + A; A \in \mathscr{L}^p_1(T^*M \otimes \mathfrak{g}_E) \} ,$$

where $\mathscr{L}^p_1(T^*M \otimes \mathfrak{g}_E)$ is the completion of $\Omega^1(\mathfrak{g}_E)$ with respect to the norm

$$\|A\|_{1,p} := \left(\int_M \|\nabla A\|^p v_g \right)^{1/p} + \left(\int_M \|A\|^p v_g \right)^{1/p} .$$

Define also the $\mathscr{L}^p$ space of G-connections of E by

$$\mathscr{L}^p(E) := \{ \nabla = \nabla^0 + A; A \in \mathscr{L}^p(T^*M \otimes \mathfrak{g}_E) \} ,$$

where $\mathscr{L}^p(T^*M \otimes \mathfrak{g}_E)$ is the completion of $\Omega^1(\mathfrak{g}_E)$ with respect to the norm

$$\|A\|_p := \left(\int_M \|A\|^p v_g \right)^{1/p} .$$

Moreover, for $0<\alpha<1$, define the α-Hölder space of G-connections of E by

$$\mathscr{C}^{\alpha}(E):=\{\nabla:=\nabla^0+A;\ A\in\mathscr{C}^{\alpha}(T^*M\otimes\mathfrak{g}_E)\},$$

where $\mathscr{C}^{\alpha}(T^*M\otimes\mathfrak{g}_E)$ is the completion of $\Omega^1(\mathfrak{g}_E)$ with respect to the norm

$$\|A\|_{\alpha}:=\inf_{x\neq y\in M}\inf_{\sigma}\inf_{\substack{X\in T_xM\\ Y\in T_yM}}\frac{\|T_{\sigma}^{-1}(A(X))-A(Y)\|}{r(x,y)^{\alpha}}.$$

Here $r(x,y)$, $x,y\in M$, is the Riemannian distance in (M,g) between x and y, σ runs through a smooth curve $[0,1]\to M$ with $\sigma(0)=x$ and $\sigma(1)=y$, and $T_{\sigma}:\mathrm{End}(E_x)\to\mathrm{End}(E_y)$ is the parallel transport with respect to the connection induced from ∇^0 along σ. We call an element in $\mathscr{C}^{\alpha}(E)$ α-Hölder continuous G-connection of E. Then the Sobolev imbedding theorem says that the imbedding $\mathscr{L}_1^p(E)\hookrightarrow\mathscr{C}^{\alpha}(E)$ is a compact operator for any $0<\alpha<1-(\dim M)/p$. Define finally our space

$$\mathscr{W}(E):=\bigcap_{p\geq 1}\mathscr{L}_1^p(E)\cap\{\nabla;\ \mathscr{YM}_e(\nabla)<\infty\}.$$

Then we obtain:

Corollary 3.2. *For any $\nabla\in\mathscr{W}(E)$, it holds that*

$$\exp\left(\frac{1}{\mathrm{Vol}(M,g)}\mathscr{YM}(\nabla)\right)\leq\frac{1}{\mathrm{Vol}(M,g)}\mathscr{YM}_e(\nabla).$$

The equality holds if and only if $\|R^{\nabla}\|$ is constant almost everywhere.

Proof. The connection ∇ satisfies $\mathscr{YM}_e(\nabla)<\infty$, i.e., $\frac{1}{2}\|R^{\nabla}\|^2$ belongs to the space $\mathscr{L}^1(M)$ of all integrable functions on M with respect to the canonical volume element v_g. Since the exponential function is convex, we get Corollary 3.2 by means of Proposition 3.1. □

Using this corollary, we obtain:

Theorem 3.3. *Assume that ∇ is a minimizer in $\mathscr{W}(E)$ of the Yang–Mills functional $\mathscr{YM}$ and the norm of the curvature $\|R^{\nabla}\|$ is almost everywhere constant. Then ∇ is also a minimizer of the exponential Yang–Mills functional $\mathscr{YM}_e$ and for any minimizer ∇' of the exponential Yang–Mills functional $\mathscr{YM}_e$ in $\mathscr{W}(E)$, the norm $\|R^{\nabla'}\|$ is almost everywhere constant.*

Proof. For any ∇' in $\mathscr{W}(E)$, it holds by definition that

$$\exp\left(\frac{1}{\operatorname{Vol}(M,g)}\mathscr{YM}(\nabla)\right)\leq \exp\left(\frac{1}{\operatorname{Vol}(M,g)}\mathscr{YM}(\nabla')\right) \tag{3.1}$$
$$\leq \frac{1}{\operatorname{Vol}(M,g)}\mathscr{YM}_e(\nabla'),$$

because of the monotonicity of the exponential function and Jensen's inequality in Proposition 3.1. Then we obtain

$$\exp\left(\frac{1}{\operatorname{Vol}(M,g)}\mathscr{YM}(\nabla)\right)\leq \inf_{\nabla'\in\mathscr{W}(E)}\frac{1}{\operatorname{Vol}(M,g)}\mathscr{YM}_e(\nabla').$$

On the other hand, since $\|R^\nabla\|$ is almost everywhere constant, we obtain:

$$\frac{1}{\operatorname{Vol}(M,g)}\mathscr{YM}_e(\nabla)=\frac{1}{\operatorname{Vol}(M,g)}\int_M \exp(\tfrac{1}{2}\|R^\nabla\|^2)v_g$$
$$=\exp(\tfrac{1}{2}\|R^\nabla\|^2)$$
$$=\exp\left(\frac{1}{\operatorname{Vol}(M,g)}\mathscr{YM}(\nabla)\right).$$

Therefore ∇ is also a minimizer of the exponential Yang–Mills functional.

Now assume that ∇' is any minimizer of the exponential Yang–Mills functional. Then the second inequality of (3.1) is in fact the equality. Due to Corollary 3.2, $\frac{1}{2}\|R^{\nabla'}\|$ is constant almost everywhere. □

4. The existence of the minimizer

By convexity of the function $\exp(\frac{1}{2}x^2)$, we shall show the exponential Yang–Mills functional is lower semi-continuous, and then by a direct method we shall show the existence of a minimizer of the exponential Yang–Mills functional.

We first prepare some results on the variational principle for a general setting following [G].

Let (M,g) be a compact Riemannian manifold. Let $F(x,y,z):M\times\mathbb{R}^N\times\mathbb{R}^m\to\mathbb{R}$ be a non-negative function satisfying the following three conditions:

(1) for all $y\in\mathbb{R}^N$ and $z\in\mathbb{R}^m$, $F(x,y,z)$ is measurable in $x\in M$,

(2) for all $z\in\mathbb{R}^m$ and almost all $x\in M$, $F(x,y,z)$ is continuous in $y\in\mathbb{R}^n$,

(3) for all $y\in\mathbb{R}^N$ and almost all $x\in M$, $F(x,y,z)$ is a convex function in z.

For two measurable functions $u: M\to\mathbb{R}^N$ and $p: M\to\mathbb{R}^m$, consider the function

$$J(u,p):=\int_M F(x,u(x),p(x))v_g(x),$$

where v_g is the canonical measure on (M,g). Then due to [G, Thm. 2.2] we get

Theorem 4.1. *Let (M, g) be a compact Riemannian manifold and $F(x, y, z): M \times \mathbb{R}^N \times \mathbb{R}^m \to \mathbb{R}$ a non-negative function satisfying the conditions (1)–(3). Assume that $\{u_i\}_{i=1}^{\infty}$ ($\{p_i\}_{i=1}^{\infty}$) be sequences of functions of M into $\mathbb{R}^N$ (of M into $\mathbb{R}^m$) satisfying that $\{u_i\}_{i=1}^{\infty}$ converges to u in $\mathscr{L}^1(M, \mathbb{R}^N)$ and $\{p_i\}_{i=1}^{\infty}$ weakly converges to p in $\mathscr{L}^1(M, \mathbb{R}^m)$. Then it holds that*

$$J(u, p) \leq \liminf_{i \to \infty} J(u_i, p_i) .$$

Due to Thm. 4.1, we have immediately

Corollary 4.2. *Let $\{\nabla_i\}_{i=1}^{\infty}$ be a sequence of connections in $\mathscr{L}_1^p(E)$ which converges weakly to a connection ∇ in $\mathscr{L}_1^p(E)$. Then it holds that*

$$\mathscr{Y}\mathscr{M}_{\mathrm{e}}(\nabla) \leq \liminf_{i \to \infty} \mathscr{Y}\mathscr{M}_{\mathrm{e}}(\nabla_i) .$$

Theorem 4.3. *The exponential Yang–Mills functional admits a minimising connection ∇ which is C^{α}-Hölder continuous for all $0 < \alpha < 1$.*

Proof. Let $\{\nabla_i = \nabla^0 + A_i\}_{i=1}^{\infty}$ be a minimising sequence of the exponential Yang–Mills functional $\mathscr{Y}\mathscr{M}_{\mathrm{e}}$ in $\mathscr{W}(E)$. Since by definition,

$$\mathscr{Y}\mathscr{M}_{\mathrm{e}}(\nabla) = \int_M \sum_{k=0}^{\infty} \frac{1}{k!} (\tfrac{1}{2} \| R^{\nabla} \|^2)^k v_g ,$$

$\{\nabla_i\}_{i=1}^{\infty}$ is bounded in $\mathscr{L}_1^p(E)$ and $\{\| R^{\nabla_i} \|\}_{i=1}^{\infty}$ is bounded in $\mathscr{L}^p(E)$ for all $1 < p < \infty$. For each $0 < \alpha < 1$ choose p with $0 < \alpha < 1 - (\dim M)/p$ to use a compact imbedding $\mathscr{L}_1^p(E) \hookrightarrow \mathscr{C}^{\alpha}(E)$. Using the compactness of the Sobolev imbedding and a diagonal argument, there exist a subsequence of $\{\nabla_i\}_{i=1}^{\infty}$, denoted by the same symbol, and a connection ∇ such that $\{\nabla_i\}_{i=1}^{\infty}$ converges weakly to ∇ in $\mathscr{L}_1^p(E)$, and converges strongly to ∇ in $\mathscr{L}^p(E)$ and $\mathscr{C}^{\alpha}(E)$. Then applying Corollary 4.2 for $\{\nabla_i\}_{i=1}^{\infty}$, we get

$$\mathscr{Y}\mathscr{M}_{\mathrm{e}}(\nabla) \leq \liminf_{i \to \infty} \mathscr{Y}\mathscr{M}_{\mathrm{e}}(\nabla_i) .$$

Therefore ∇ attains a minimum of $\mathscr{Y}\mathscr{M}_{\mathrm{e}}$ and belongs to $\mathscr{Y}\mathscr{M}_{\mathrm{e}}$ and belongs to $\mathscr{C}^{\alpha}(E)$. □

5. Existence of exponential Yang–Mills connections

In this section, we show existence of Yang–Mills connections and exponential Yang–Mills connections. For the existence of Yang–Mills connections, we have:

Theorem 5.1 (Katagiri [KA]). *Let (M, g) be an n-dimensional Riemannian manifold, G a compact Lie group and E a G-vector bundle over M. Assume that $n \geq 5$. Then there exist a C^∞ Riemannian metric $\tilde{g}$ on M conformal to g and a C^∞ G-connection ∇ on E such that ∇ is a Yang–Mills connection with respect to $\tilde{g}$.*

Remark. In the case $n=4$, the Yang–Mills functional $\mathscr{YM}$ is invariant under the change g to $\tilde{g}=fg$ with a positive C^∞ function f of M. In the case $n=2$ or 3, Yang–Mills connections exist for any G-vector bundle E over any Riemannian manifold (M, g) of dimension n (cf. [R]).

Proof. For completeness, we give here a brief proof. For a positive C^∞ function f on M, put a new Riemannian metric $\tilde{g}$ on M by $\tilde{g}=fg$. We write the subscripts g and $\tilde{g}$ for their corresponding quantities. Then we get

$$\int_M \|R^\nabla\|_{\tilde{g}}^2 v_{\tilde{g}} = \int_M f^{(n-4)/2} \|R^\nabla\|_g^2 v_g .$$

For the Euler–Lagrange equation,

$$\delta_{\tilde{g}}^\nabla R^\nabla = 0 \quad \Leftrightarrow \quad \delta_g^\nabla (f^{(n-4)/2} R^\nabla) = 0 ,$$

where δ_g^∇, $\delta_{\tilde{g}}^\nabla$ are the formal adjoints of d^∇ corresponding to g and $\tilde{g}$, respectively.

Moreover, the functional

$$F_p(\nabla) := \frac{1}{2} \int_M (1 + \|R^\nabla\|_g^2)^{p/2} v_g$$

satisfies the Palais–Smale condition and attains a minimum if $2p > \dim M$ (cf. [Uh, P]). Its Euler–Lagrange equation is given by

$$\delta_g^\nabla ((1 + \|R^\nabla\|_g^2)^{(p-2)/2} R^\nabla) = 0 . \tag{5.1}$$

In fact, for $A \in \Omega^1(\mathfrak{g}_E)$,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} F_p(\nabla + tA) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \int_M (1 + \|R^{\nabla + tA}\|_g^2)^{p/2} v_g$$

$$= \frac{p}{2} \int_M (1 + \|R^\nabla\|_g^2)^{(p-2)/2} \langle \mathrm{d}^\nabla A, R^\nabla \rangle_g v_g .$$

Thus eq. (5.1) has a solution ∇ for $2p > \dim M$. For the solution ∇, defining

$$f := (1 + \|R^\nabla\|_g^2)^{(p-2)/(n-4)} ,$$

and $\tilde{g}=fg$, we obtain $\delta_{\tilde{g}}^\nabla \mathrm{R}^\nabla = 0$ so $\tilde{g}$ and ∇ are the desired ones. □

Now our theorem is as follows:

Theorem 5.2. *Let (M, g) be an n-dimensional compact Riemannian manifold, G a compact Lie group, and E a G-vector bundle over M. Assume that $n \geq 5$. Then there exist a C^∞ Riemannian metric $\tilde{g}$ on M which is conformal to g and a C_∞ G-connection ∇ on E such that ∇ is an exponential Yang–Mills connection with respect to $\tilde{g}$.*

Thm. 5.2 follows immediately from Thm. 5.1 and the following theorem:

Theorem 5.3. *Under the same situation of Thm. 5.2, assume that $n \geq 5$ and let ∇ be a Yang–Mills connection. Then there exists a C^∞ Riemannian metric on M which is conformal to g, such that ∇ is an exponential Yang–Mills connection with respect to $\tilde{g}$.*

To prove this theorem, we need the following two lemmas.

Lemma 5.4. *The function $f \mapsto \log f / f^2$ is a strictly increasing function on the interval $[1, \sqrt{e})$. Thus the inverse function $f = \Phi(y)$ exists on the interval $[0, 1/2e)$ and smooth.*

Proof. In fact, the derivative is

$$\frac{dy}{df} = \frac{1 - 2\log f}{f^3},$$

which is positive on the interval $[1, \sqrt{e})$. □

Lemma 5.5. *Under the same situation of Thm. 5.2, assume that $n \geq 5$ and ∇ is a Yang–Mills connection. Then for any $\epsilon > 0$, there exists a C^∞ Riemannian metric $\tilde{g}$ on M which is homothetic to g such that ∇ is a Yang–Mills connection with respect to $\tilde{g}$ and $\|R^\nabla\|^2_{\tilde{g}} < \epsilon$.*

Proof. For a positive constant C, put $\tilde{g} = C\,g$. Then the Yang–Mills equation for $\tilde{g}$ is the same as for g. Moreover, since $\|R^\nabla\|^2_{\tilde{g}} = C^{-2}\|R^\nabla\|^2_g$ and M is compact, we get $\|R^\nabla\|^2_g < \epsilon$ if C is sufficiently large. □

Proof of Theorem 5.3. By Lemma 5.5, we may assume a Yang–Mills connection ∇ satisfies $\|R^\nabla\|^2 < \epsilon < (n-4)/2e$. For a positive C^∞ function f on M, define $\tilde{g} = f^{-1} g$. Then

$$\delta^\nabla_{\tilde{g}} R^\nabla = 0 \quad \Leftrightarrow \quad \delta^\nabla_g (f^{(n-4)/2} R^\nabla) = 0\,.$$

Since $\|R^\nabla\|^2_g < (n-4)/2e$, we can define the function f on M by

$$f := \Phi(\,\|R^\nabla\|^2_g/(n-4)\,) > 0\,,$$

due to Lemma 5.4. Then it holds that

$$f^{(n-4)/2} = (\exp(f^2 \|R^{\nabla}\|_g^2/(n-4)))^{(n-4)/2}$$
$$= \exp(f^2 \|R^{\nabla}\|_g^2/2) = \exp(\|R^{\nabla}\|_{\tilde{g}}^2/2) .$$

Then it holds that

$$\delta_{\tilde{g}}^{\nabla}(\exp(\tfrac{1}{2}\|R^{\nabla}\|_{\tilde{g}}^2)R^{\nabla}) = 0 ,$$

which implies that ∇ is an exponential Yang–Mills connection with respect to $\tilde{g}$. □

For the case $n = \dim M = 4$, we obtain the following:

Theorem 5.6. *Let (M, g) be a 4-dimensional compact Riemannian manifold. Let G be a compact Lie group and E be a G-vector bundle over M. Then there exist a C^0 (continuous) Riemannian metric on M which is conformal to g, and a C^∞ G-connection ∇ such that ∇ is an exponential Yang–Mills connection in the weak sense.*

Remark. Here an exponential Yang–Mills connection ∇ *in the weak sense* means that a C^0 Riemannian metric $\tilde{g}$ and C^∞ G-connection ∇ satisfy that

$$\int_M \langle d^{\nabla}A, \exp(\tfrac{1}{2}\|R^{\nabla}\|_{\tilde{g}}^2)R^{\nabla}\rangle_{\tilde{g}} v_{\tilde{g}} = 0 ,$$

for all $A \in \Omega^1(\mathfrak{g}_E)$.

Proof. We first note the conformal changes of the exponential Yang–Mills functional and the equation of the exponential Yang–Mills connections. For any n-dimensional Riemannian manifold (M, g) and any positive C^∞ function f on M, put $\tilde{g} := fg$. Then the corresponding exponential Yang–Mills functional is

$$\mathcal{YM}_{e,\tilde{g}} := \int_M \exp(\tfrac{1}{2}\|R^{\nabla}\|_{\tilde{g}}^2) v_{\tilde{g}} = \int_M f^{n/2} \exp(\tfrac{1}{2} f^{-2}\|R^{\nabla}\|_g^2) v_g .$$

The Euler–Lagrange equation is given as follows: for any $A \in \Omega(\mathfrak{g}_E)$,

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{YM}_{e,\tilde{g}}(\nabla + tA) = \int_M f^{(n-4)/2} \exp(\tfrac{1}{2} f^{-2}\|R^{\nabla}\|_g^2) \langle d^{\nabla}A, R^{\nabla}\rangle_g v_g.$$

Therefore we get

$$\delta_{\tilde{g}}(\exp(\tfrac{1}{2}\|R^{\nabla}\|_{\tilde{g}}^2)R^{\nabla}) = 0 \quad \Leftrightarrow \quad \delta_g^{\nabla}(f^{(n-4)/2} \exp(\tfrac{1}{2} f^{-2}\|R^{\nabla}\|_g^2)R^{\nabla}) = 0.$$

In the case $\dim M = 4$, the corresponding Euler–Lagrange equation to $\tilde{g}$ is

$$\delta_g^{\nabla}(\exp(\tfrac{1}{2} f^{-2}\|R^{\nabla}\|_g^2)R^{\nabla}) = 0 .$$

Remember that a C^∞ solution of Eq. (5.1),

$$\delta_g^\nabla((1+\|R^\nabla\|_g^2)^{(p-2)/2}R^\nabla)=0,$$

exists for $p>2$ in the case of dim $M=4$. For the solution ∇, we can define a C^0 function f on M by

$$f=\begin{cases}\sqrt{\dfrac{\|R^\nabla\|_g^2}{\log((1+\|R^\nabla\|_g^2)^{p-2})}}, & \|R^\nabla\|\neq 0,\\[2ex] \sqrt{\dfrac{1}{p-2}}, & \|R^\nabla\|=0,\end{cases}$$

and put $\tilde{g}:=fg$. Then it holds that for any $A\in\Omega^1(\mathfrak{g}_E)$,

$$\int_M \langle d^\nabla A, \exp(\tfrac{1}{2}\|R^\nabla\|_{\tilde{g}}^2)R^\nabla\rangle_{\tilde{g}} v_{\tilde{g}}=0.$$

We obtain Thm. 5.6. □

6. The second variation formula

In this section, we calculate the second variation of the exponential Yang–Mills functional. The calculation is similar as in Bourguignon and Lawson [BL].

We retain the above situation: Let (M,g) be an n dimensional compact Riemannian manifold, G a compact Lie group and E a G-vector bundle over M. We suppose that ∇^t, $|t|<\epsilon$, is a smooth family of G-connections on E where $\nabla=\nabla^0$ is an exponential Yang–Mills connection. We write

$$\nabla^t=\nabla+A^t,$$

where $A^t\in\Omega^1(\mathfrak{g}_E)$ for all t, and $A^0=0$. The infinitesimal variation of the connection associated to ∇^t at $t=0$ is

$$B:=\left.\frac{dA^t}{dt}\right|_{t=0}\in\Omega^1(\mathfrak{g}_E).$$

Define an endomorphism $\mathfrak{R}^\nabla$ of $\Omega^1(\mathfrak{g}_E)$ following [BL] by

$$\mathfrak{R}^\nabla(\varphi)(X):=\sum_{j=1}^n[R^\nabla(e_j,X),\varphi(e_j)],$$

for $\varphi\in\Omega^1(\mathfrak{g}_E)$, where $\{e_j\}_{j=1}^n$ is a local orthonormal frame field of (M,g). Then we obtain:

Theorem 6.1. *Let (M, g) be an n dimensional compact Riemannian manifold, G a compact Lie group and E a G-vector bundle over M. Let ∇ be an exponential Yang–Mills connection on E. Then the second variation of the exponential Yang–Mills functional is given by*

$$\frac{d^2}{dt^2}\bigg|_{t=0} \mathcal{YM}_e(\nabla^t) = \int_M \exp(\tfrac{1}{2}\|R^{\nabla}\|^2)\{\langle d^{\nabla}B, R^{\nabla}\rangle^2 + \langle d^{\nabla}B, d^{\nabla}B\rangle + \langle B, \mathfrak{R}^{\nabla}(B)\rangle\}v_g$$

$$= \int_M \langle \mathcal{S}^{\nabla}(B), B\rangle v_g ,$$

for $B := (d/dt)|_{t=0}\nabla^t \in \Omega^1(\mathfrak{g}_E)$, where $\mathcal{S}^{\nabla}$ is a differential operator acting on $\Omega^1(\mathfrak{g}_E)$ defined by

$$\mathcal{S}^{\nabla}(B) := \delta^{\nabla}(\exp(\tfrac{1}{2}\|R^{\nabla}\|^2)\langle d^{\nabla}B, R^{\nabla}\rangle R^{\nabla}) + \delta^{\nabla}(\exp(\tfrac{1}{2}\|R^{\nabla}\|^2)d^{\nabla}B) + \exp(\tfrac{1}{2}\|R^{\nabla}\|^2)\mathfrak{R}^{\nabla}(B) .$$

Proof. In fact, we immediately obtain:

$$\frac{1}{2}\frac{d^2}{dt^2}\bigg|_{t=0} \|R^{\nabla^t}\|^2 = \langle d^{\nabla}B, d^{\nabla}B\rangle + \langle d^{\nabla}C + [B\wedge B], R^{\nabla}\rangle ,$$

where $C := (d^2/dt^2)|_{t=0}\nabla^t$. Thus we obtain

$$\frac{d^2}{dt^2}\bigg|_{t=0} = \frac{d}{dt}\bigg|_{t=0} \int_M \tfrac{1}{2}\exp(\tfrac{1}{2}\|R^{\nabla^t}\|^2)\frac{d}{dt}\|R^{\nabla^t}\|^2 v_g$$

$$= \frac{1}{4}\int_M \exp(\tfrac{1}{2}\|R^{\nabla}\|^2)\left\{\left(\frac{d}{dt}\bigg|_{t=0}\|R^{\nabla^t}\|^2\right)^2 + 2\frac{d^2}{dt^2}\bigg|_{t=0}\|R^{\nabla^t}\|^2\right\}v_g$$

$$= \int_M \exp(\tfrac{1}{2}\|R^{\nabla}\|^2)\{\langle d^{\nabla}B, R^{\nabla}\rangle^2 + \langle d^{\nabla}C + [B\wedge B], R^{\nabla}\rangle + \langle d^{\nabla}B, d^{\nabla}B\rangle\}v_g .$$

Furthermore, since ∇ is an exponential Yang–Mills connection,

$$\int_M \exp(\tfrac{1}{2}\|R^{\nabla}\|^2)\langle d^{\nabla}C, R^{\nabla}\rangle v_g = \int_M \langle C, \delta^{\nabla}(\exp(\tfrac{1}{2}\|R^{\nabla}\|^2)R^{\nabla})\rangle v_g = 0 .$$

And we know (cf. (6.7) in [BL])

$$\langle [B\wedge B], R^{\nabla}\rangle = \langle B, \mathfrak{R}^{\nabla}(B)\rangle .$$

Thus we obtain the desired formula. □

The index, nullity and stability of an exponential Yang–Mills connection ∇ can be defined in the same way as in the case of Yang–Mills connections due to Thm. 6.1. But it is rather difficult to analyse them because the form of $\mathscr{S}^{\nabla}$ is much more complicated compared with the case of Yang–Mills connections.

Here we only note the case that $\|R^{\nabla}\|$ is constant. We immediately obtain:

Corollary 6.2. *Let ∇ be an exponential Yang–Mills connection of which $\|R^{\nabla}\|$ is constant. Then the stability as a Yang–Mills connection implies the stability as an exponential Yang–Mills connection.*

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